Imperial College London

Lecture 11

Discrete Time Signals

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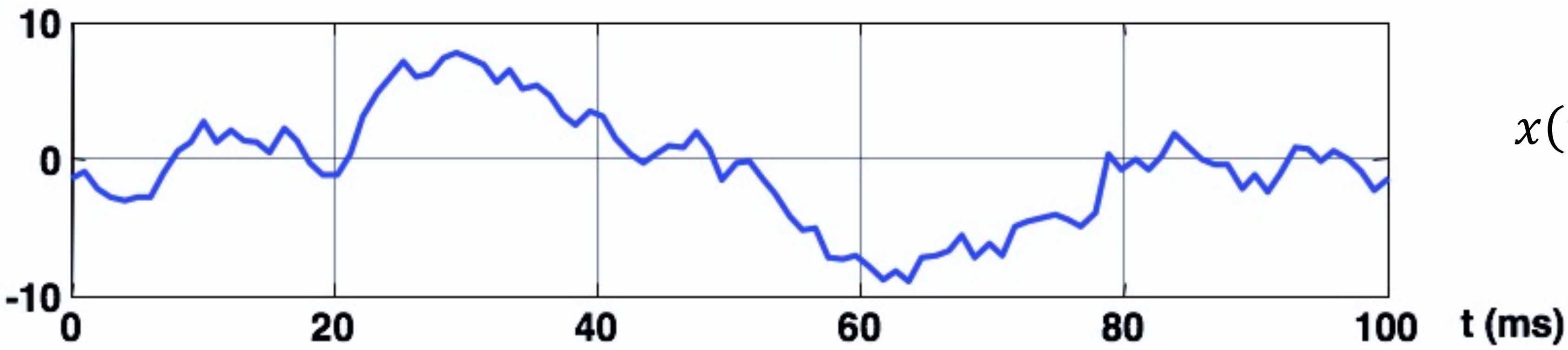
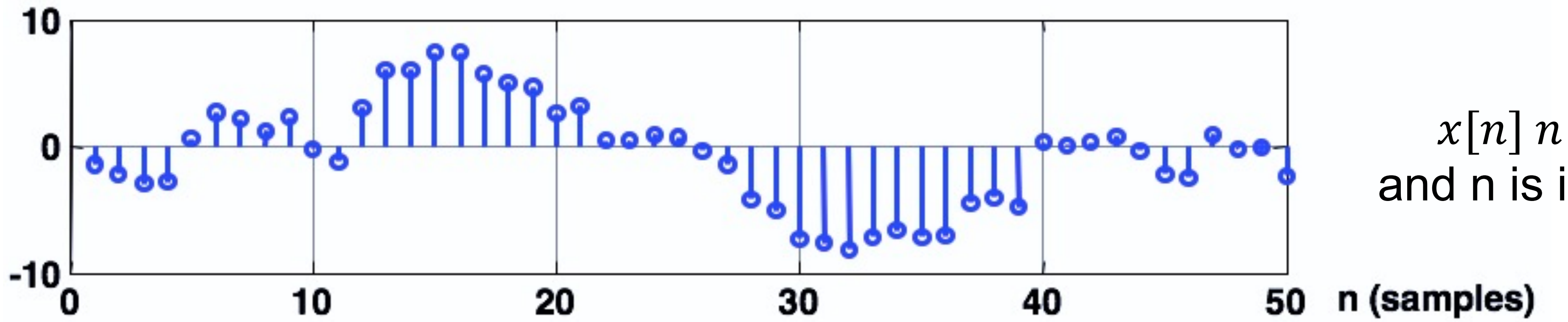
Discrete time signals

- Sampling a continuous time signal at frequency fs converts the signal to discrete time.
- How frequently we need to sample is governed by the Sampling Theorem.



$x(t)\ t \geq 0$

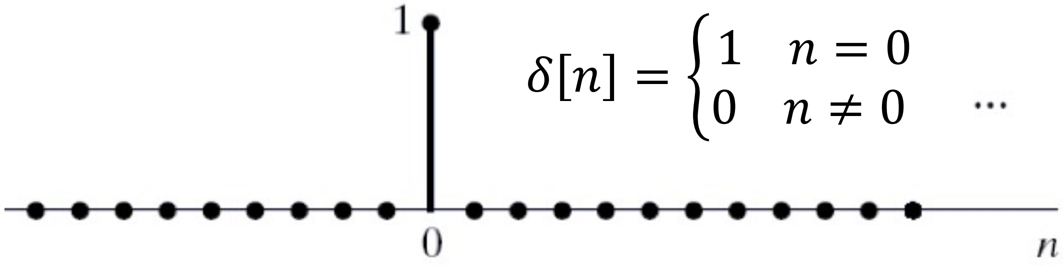


$x[n]\ n \geq 0$
and n is integer

Basic discrete signals

Impulse signal

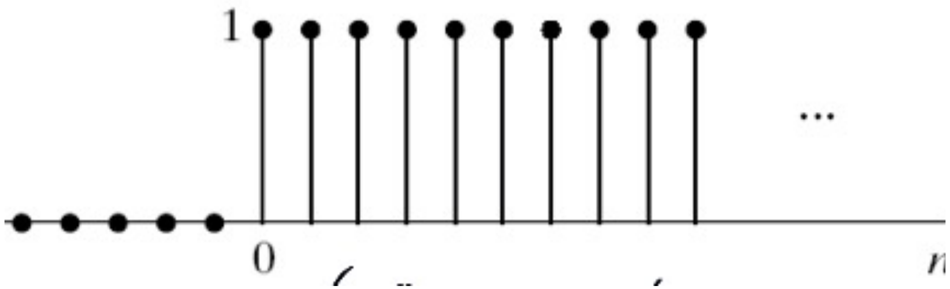
$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



$$x[n] = A,$$
$$\text{for all n} = \text{integer}$$

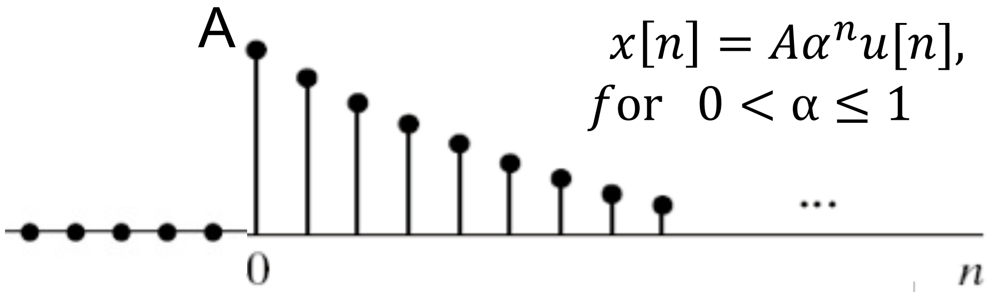
Unit step signal

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

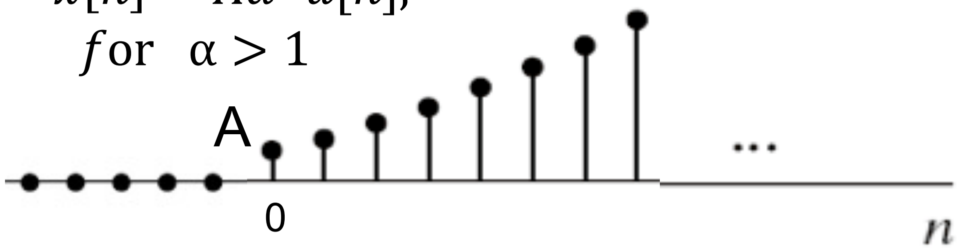


Causal exponential signals

$$x[n] = A\alpha^n u[n],$$
$$for \quad 0 < \alpha \leq 1$$

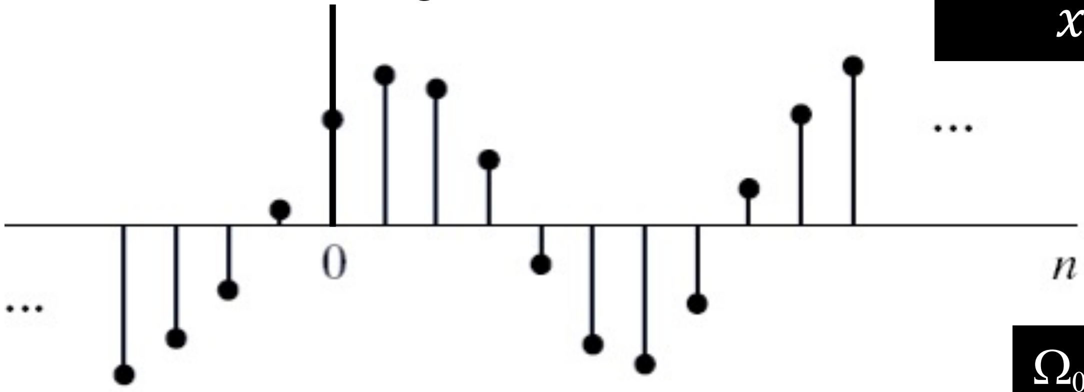


$$x[n] = A\alpha^n u[n],$$
$$for \quad \alpha > 1$$



Discrete sinusoidal signal

Cosine signal



$x[n] = A\cos(\Omega_0 n + \phi)$ for n = integer

Ω_0 is angle increment between samples
In radian/sample

- Compare this with continuous time signal equation:

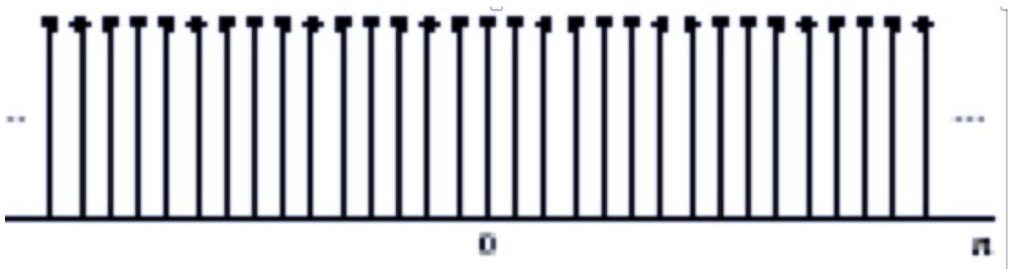
$$x(t) = A\cos(\omega_0 t + \phi) \xrightarrow{\text{sampling}} x[n] = A\cos(\Omega_0 n + \phi)$$

- The discrete time signal is sampled at f_s, where $T_s = 1/f_s$ is the sampling period (i.e. time step between successive samples).
- Note that Ω_0 in discrete time domain is angle increment of this sinusoidal signal between samples. Its unit is radians/sample (not rad/sec as in continuous time case.

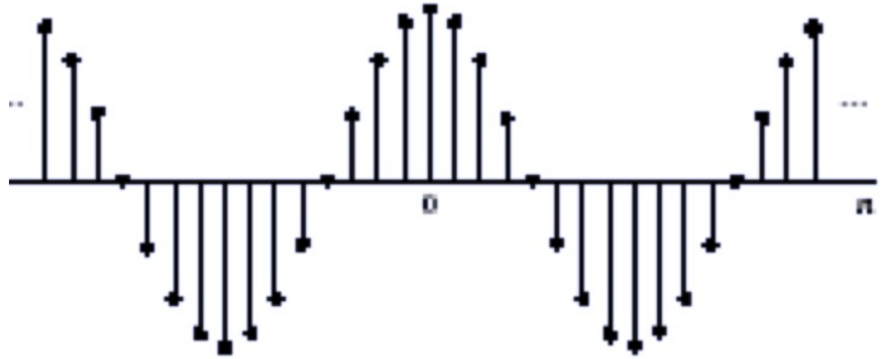
Discrete sinusoidal signal

$x[n] = A\cos(\Omega_0 n + \phi) \quad \text{for n = integer}$

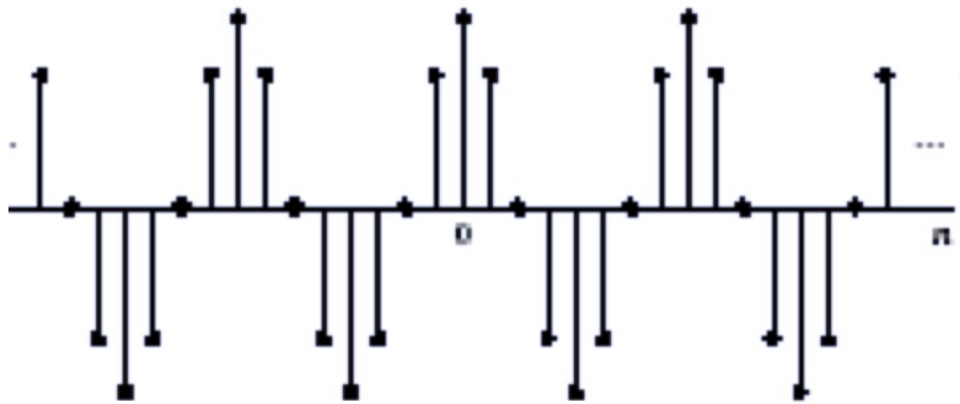
Ω_0 is in rad/sample



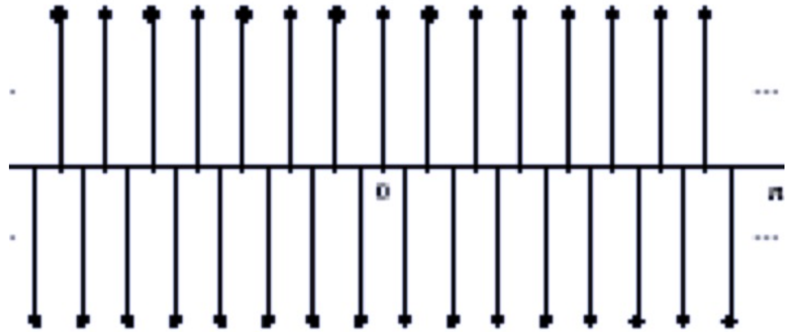
(a) When $\Omega_0 = 0$ *or* $x[n]$ is DC



(b) When $\Omega_0 = \frac{\pi}{8}$
i.e. 16 sample/cycle



(c) When $\Omega_0 = \frac{\pi}{4}$
i.e. 8 sample/cycle



(d) When $\Omega_0 = \pi$
i.e. 2 sample/cycle

Operations on discrete signals

- Sum of two signals:

$$s[n] = x[n] + y[n]$$

- Product of two signals:

$$p[n] = x[n] \,.\, y[n]$$

- Amplification of a signal:

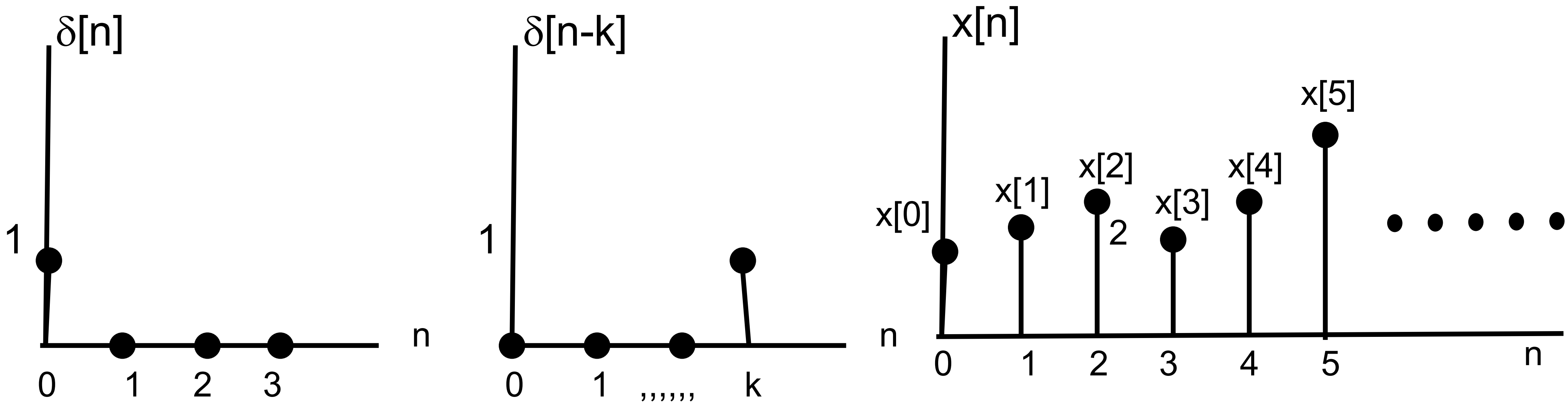
$$y[n] = \alpha \,.\, x[n]$$

- Delaying a signal by k samples:

$$y[n] = x[n-k]$$

Discrete signal and impulses

- We can represent a causal discrete signal $x[n]$ in terms of sum of weighted delayed impulses:

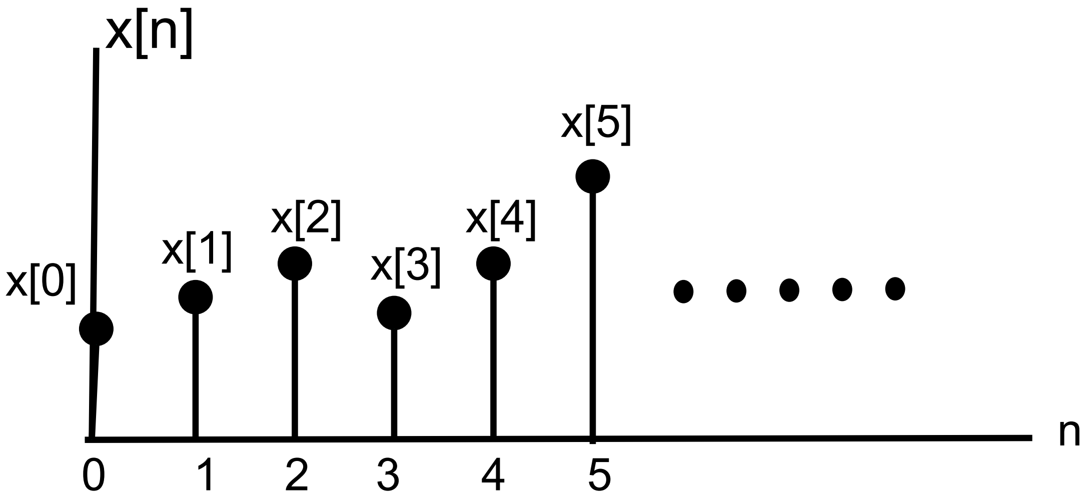


$$x[n] = x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[2]\,\delta[n-2] + x[3]\,\delta[n-3] + \ldots\ldots$$

$$x[n] = \sum_{k=0}^{\infty} x[k]\,\delta[n-k]$$

Energy of a discrete signal

- The energy of a discrete signal can be computed easily – simply sum the square of each sample values:



$$E\{x[n]\} = \sum_{k=0}^{\infty} |x[k]|^2$$

- Instantaneous energy of the signal at sample *i* over a window of *K* samples is:

$$E\{x_i[n]\} = \sum_{k=0}^{K-1} |x[i+k]|^2$$

An alternative representation of discrete signals

- Instead of representing discrete signals in terms of impulse functions with various delay, we can transform the discrete signal into another domain (or mathematical representation).
- Let us assume that we use a transformation that maps an impulse function with delay k such that:

$$\delta[n-k] \overset{Z}{\rightarrow} z^{-k}$$

- Then the discrete signal x[n] is transformed to another function in terms of the variable z:

$$x[n] = x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[2]\,\delta[n-2] + x[3]\,\delta[n-3] + \ldots\ldots$$

$$x[n] \overset{Z}{\rightarrow} X[z] = x[0]\,z^{0} + x[1]z^{-1} + x[2]z^{-1} + x[3]z^{-3} + \ldots\ldots$$

$$X[z] = \sum_{k=0}^{\infty} x[k]z^{-k}$$

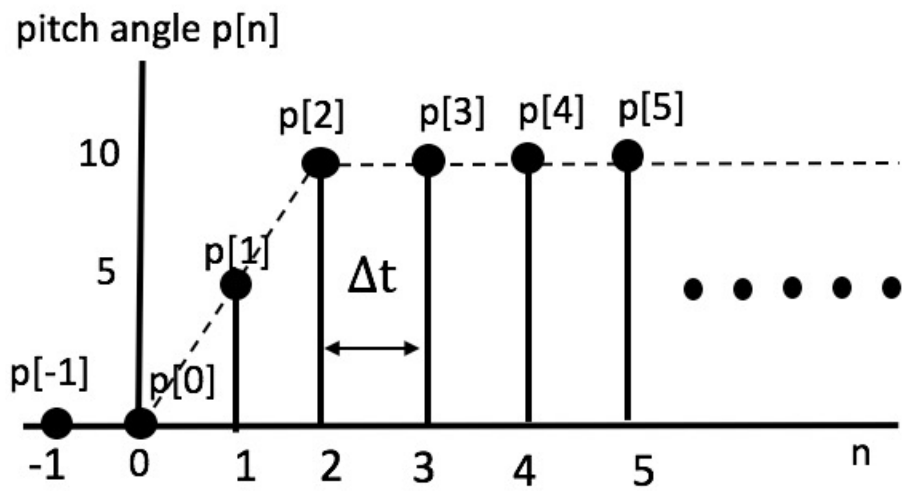
- X[z] is the z-transform of the signal *x[n]*. For now, you only need to remember that z^{-k} represents k sample period delay.

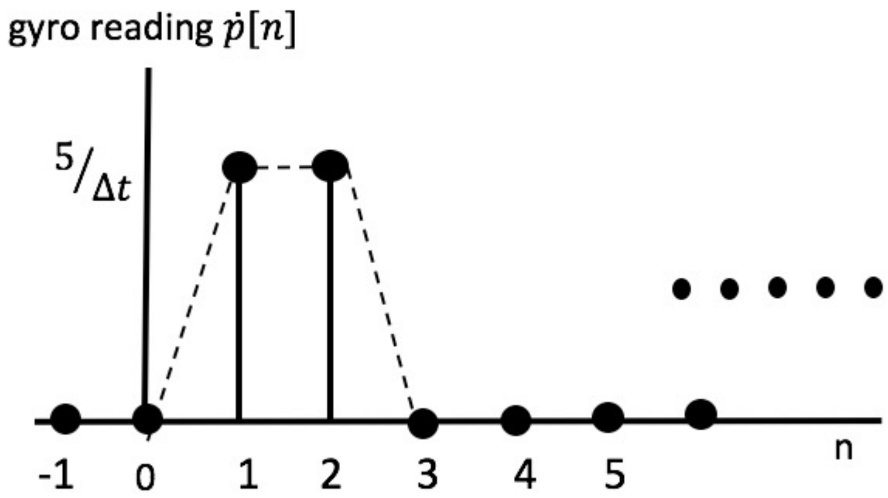
Example – Gyroscope signal

- Assume pitch angle p(t) changes from 0 to 10 shortly after t = 0
- After sampling, we get p[n] as shown where it takes two sample periods to reach final value of 10
- Gyroscope measure angular velocity $\frac{dp}{dt}$
- In discrete time domain, we get

$$\frac{\Delta p}{\Delta t} = \dot{p}[n] = \frac{p[n] - p[n-1]}{\Delta t}$$

- For discrete signals, we compute differentiation using differences
- This graph shows the sample values of the gyroscope reading



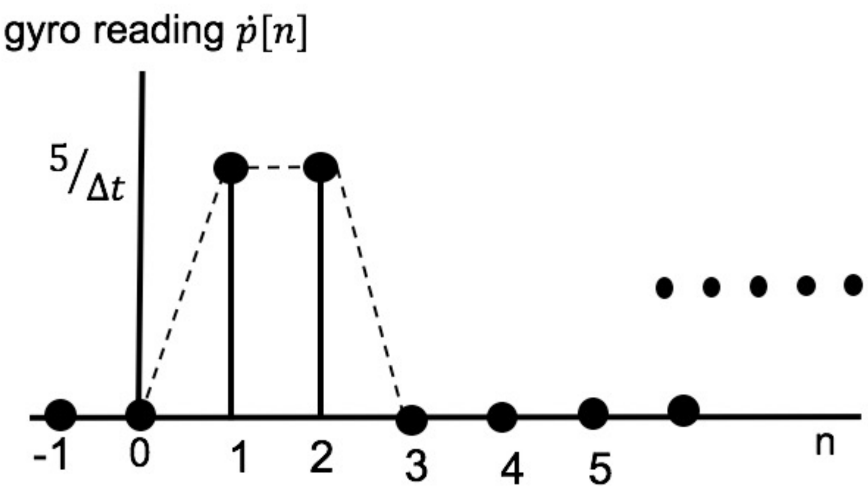


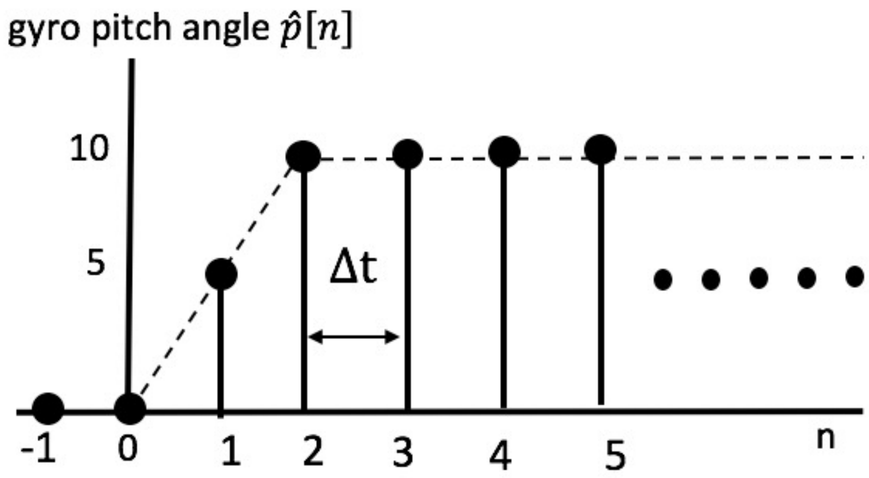
Derive pitch angle from gyro data

- How do we obtain an estimate of the pitch angle $\hat{p}[n]$ from the gyro reading $\dot{p}[n]$?
- We perform integration:

$$\hat{p}[\mathrm{n}] = \hat{p}[\mathrm{n}-1] + \dot{p}[n]\Delta t$$

- So integration in discrete time domain is perform with summation in a **recursive equation**

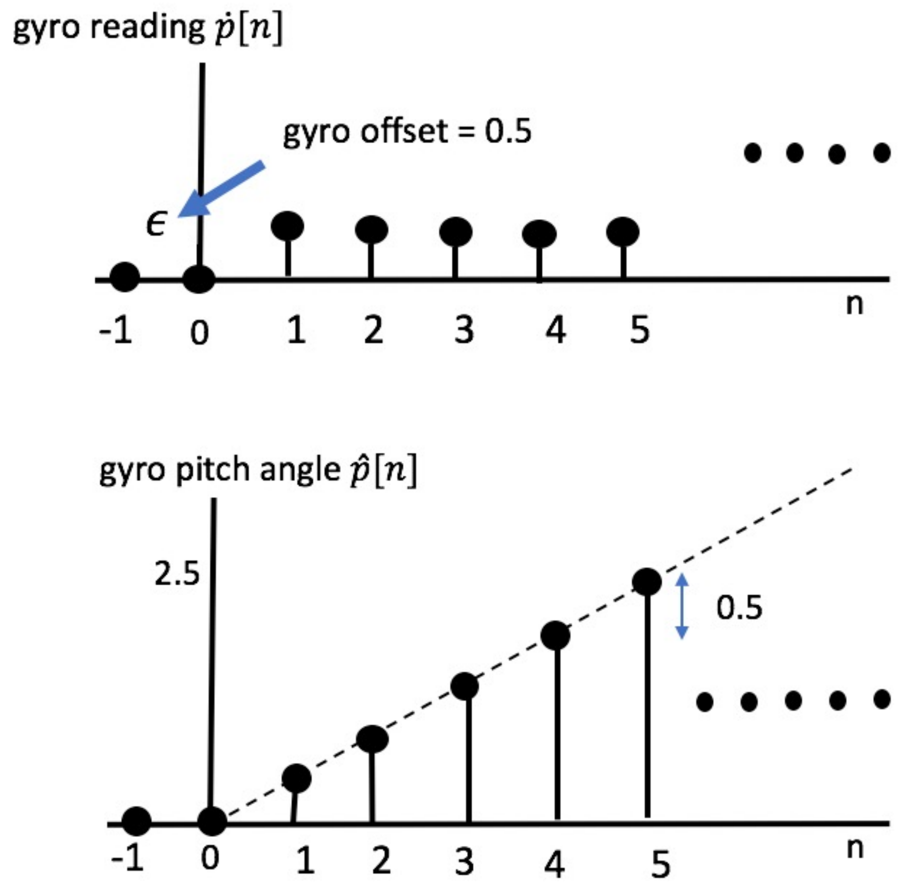




Problem of Drift in Gyroscope

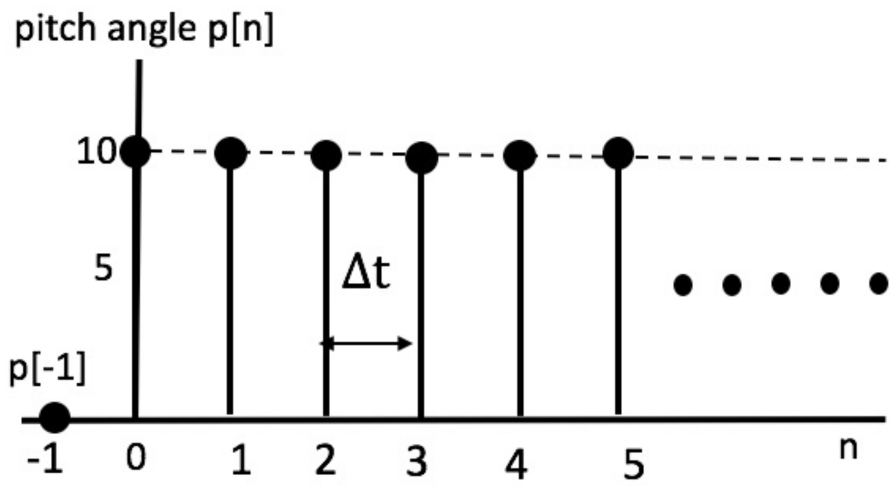
- All transducers that measure physical quantities (such as angular velocity) has errors
- In gyroscope, the problematic error is the DC offset. That means even if the gyro is NOT rotating, the IMU returns a small value ϵ
- The result of such offset after integration is to yield a pitch angle estimate that increases or decrease linearly with time as shown here.

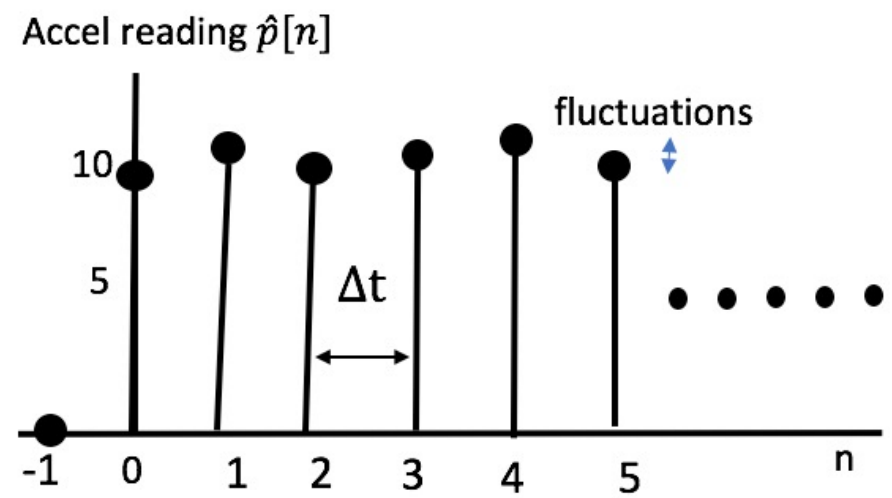
$$\hat{p}[\mathrm{n}] = \hat{p}[\mathrm{n}-1] + \varepsilon \quad \text{for } \mathrm{n} > 0$$



Problem of noise in accelerometer data

- Let us assume that the Pybench board is instantaneously rotated to 10 degrees
- The accelerometer's measurement will reflect this, but superimpose on this is a lot of noise
- Partly is due to unwanted forces acting on the mass inside the IMU





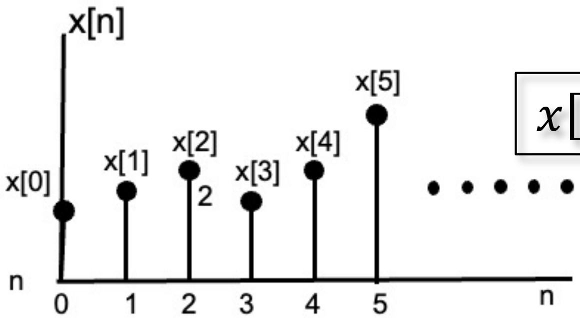
Three Big Ideas (1)

1. Discrete sinusoidal signal is of the form:

$$x(t) = A\cos(\omega_0 t + \phi) \xrightarrow{\text{sampling}} x[n] = A\cos(\Omega_0 n + \phi)$$

Ω_0 is angle increment between samples
In radian/sample

2. Any discrete signal can be expressed as **weighted sum** of unit impulses, delayed and scaled.



$$x[n] = x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[2]\,\delta[n-2] + x[3]\,\delta[n-3] + \ldots\ldots$$

Three Big Ideas (2)

3. If we map the delayed impulse (delay function) as:

$$\delta[n-k] \;\rightarrow\; z^{-k}$$

We transforms discrete time signals to an new domain, called z-domain.

This transform is known as z-transform, and is useful to model discrete signals.

$$x[n] \overset{\text{z-transform}}{\rightarrow} X[z] = x[0]\,z^{0} + x[1]z^{-1} + x[2]z^{-1} + x[3]z^{-3} + \ldots\ldots$$